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An Implicit Class of Robust Control System Characterized by a Convex-Type Matrix Constraints

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***Abstract*—Considering the use of the so-called *Composite Quadratic Lyapunov Function* (CQLF) based on descriptor techniques from the stability theory of continuous systems, this paper designs a linear feedback control to deal with bounded uncertainties in a class of nonlinear dynamical system described by semi-explicit *Differential Algebraic Equations* (DAE) by application of an extended version of the conventional *Attractive Invariant Ellipsoid Method* (AEM). The resulting closed-loop system converges to a minimal size set generated by the convex hull of some ellipsoids. The use of CQLF in the modified AEM allows obtaining an appropriate feedback control to ensure robustness and stability properties of the system. An academic example is presented to illustrate the applicability of the contribution.**

***Index Terms*—Robust Control, DAE, Composite Quadratic Lyapunov Functions, Attractive Ellipsoid, convex-type matrix constraints, Implicit Systems**

I. Introduction

Many processes in control theory, can be described, at least approximately by the nonlinear dynamical system of control:

$$\begin{aligned} \dot{x} &= \phi(x(t), u(t), t) \\ y &= g(x(t), t), \end{aligned} \tag{1}$$

where functions $\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{I} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{I} \to \mathbb{R}^p$, are unknown but they belong to the given *Quasi-Lipschitz* (Q-L) classes $\mathcal{C}_\phi$ and $\mathcal{C}_g$ of nonlinear functions, respectively, to ensure solution of system (1) through AEM here studied. And $\mathbb{I} \subseteq \mathbb{R}_+$. The *input* or *control* $u(t)$ is chosen from a control set $U \subset \mathbb{R}^m$ of admissible control functions of type $u(x)$ such that the closed-loop system has a well-defined solution. The functions in the given Q-L class are limited as in the *Lipschitz* function in how fast they change, i.e. the whole graph always remains outside the region limited by the *double parabolic cone*. The linearization-like idea of AEM allows transforming a nonlinear dynamic system of control with a Q-L right-hand side into an equivalent linear system. This linearization-like idea is to compatible with several widely used techniques of linear approximation related to plant models and is common in the theoretical and numerical practice of control engineering e.g., [1]. This class of systems has been the object of the AEM, where system (1) is described by *Ordinary Differential Equations* (ODE) e.g., [2]–[5].

Most physical plants are covered by this system class but there exist systems which do not fit into (1). There exists an easier way to model it in a more general framework in order to keep the natural structure of the problem, where the ODE is represented by $F(\dot{z}(t), z(t), u(t)) = 0$, where F may be any vector valued function. If F is a function with singular $F_{\dot{z}}$, where $F_{\dot{z}}$ denotes $\partial F(\dot{z}(t), z(t), u(t))/\partial \dot{z}$, the system with $F_{\dot{z}}$ singular is called a DAE because it can include both differential and algebraic equations of the variables. The vector $z(t)$ is called the generalized state vector or descriptor vector and the components are called generalized states or descriptor variables, see the contribution of Bender and Laub in [6]. Modeling with DAE plays a vital role for constrained mechanical systems, electrical circuits, chemical reaction kinetics and others. Those readers who would like to know more about the topic of DAE and the accumulated work over the years, please see [7]–[10]. The DAE class studied in this contribution is called semi-explicit DAE (see [7], [8] for more details). One accepted form to describe a semi-explicit DAE is:

$$\begin{aligned} \dot{z}_1 &= \iota(z_1, z_2) \in \mathbb{R}^m \\ 0 &= \varrho(x_1, x_2) \in \mathbb{R}^n, \end{aligned} \tag{2}$$

where $z := (z_1, z_2) \in \mathbb{R}^m \times \mathbb{R}^n$, $\iota : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $\varrho : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and the partial derivative $\partial\varrho/\partial z_2$ has a bounded inverse in a neighborhood of the solution. Assuming we have a set of consistent initial values (z_{10}, z_{20}) it follows from the inverse function theorem that z_2 can be found as a function of z_1. Thus local existence, uniqueness and regularity of the solution follows from the conventional theory of ODE.

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The concept of *differentiation index* along the solution path is defined as the minimum number of times that all or part of $F(z, \dot{z}) := (\dot{z}_1 - \iota(z_1, z_2), \varrho(z_1, z_2))^\intercal$ must be differentiated with respect to t in order to determine $\dot{z}$ as a continuous function of t and z, where $\dot{z} := (\dot{z}_1, \dot{z}_2) \in \mathbb{R}^m \times \mathbb{R}^n$. A semi-explicit DAE has an index one and eventually is reduced to an ODE on a manifold of z_1. Then ODE in system (1) is replaced by semi-explicit DAE: $F(z, \dot{z}) = 0$. The Q-L linearization-like approximation allows to rewrite system $F(z, \dot{z}) = 0$ into a linear control problem with variable coefficients in the form:

$$\begin{aligned} E(t)\dot{z} &= A(t)z + B(t)u(t) + \zeta(t) \\ y &= C(t)z + \kappa(t), \end{aligned} \tag{3}$$

where $g(z(t), t) := C(t)z + \kappa(t)$; E, $A : \mathbb{I} \to \mathbb{R}^{n\times n}$, $B : \mathbb{I} \to \mathbb{R}^{n\times n}$, $C : \mathbb{I} \to \mathbb{R}^{q\times n}$ are matrix functions, with $\det(E) \neq 0$, $\forall t \in \mathbb{I}$. Functions $\zeta \in \mathcal{C}_\zeta(\mathbb{I}, \mathbb{R}^n)$, $\kappa \in \mathcal{C}_\kappa(\mathbb{I}, \mathbb{R}^q)$, are nonlinear and bounded vector functions for some interval $\mathbb{I} \in \mathbb{R}_+$. Vector $z \in \mathbb{R}^n$ represents the generalized state, $u \in \mathbb{R}^m$ the input or control, and $y \in \mathbb{R}^q$ the output of the system, see for example [11]. All properties of the *descriptor system* $F(\dot{z}(t), z(t), u(t)) = 0$ can be determined by computing the *invariants* of the associated matrix pair $(E(t), A(t))$ under equivalence transformations. This means that we can consider transformed problem (3) instead of original problem $F(\dot{z}(t), z(t), u(t)) = 0$ with respect to solvability and related questions. *Regularity* of previous matrix pair is closely related to the solution behavior of the corresponding DAE. A completely algebraic characterization for the solution of system (3) is possible when system is regular and consistent for all $t \in \mathbb{I}$. The control problem (3) is called *consistent* if there exists an input function $u(t)$, for which the resulting DAE is solvable. The control problem is also called *regular* if for every sufficiently smooth input function $u(t)$ and inhomogeneity ζ the corresponding DAE is solvable and the solution is unique for every consistent initial value and for all $t \in \mathbb{I}$. But, unfortunately, regularity and consistency conditions do not guarantee unique solvability of the initial value problem. A general example of this approach please read [7]. Then, the control $u(t)$ is designed as a feedback of a given structure containing a convex bounded set of parameters $\mathcal{E}(P, \rho) := \{z \in \mathbb{R}^n | z^\intercal P^{-1} z \leq \rho\}$, where P is a symmetric positive definite $n \times n$ matrix. The AEM suggests that we select the feedback parameters $\mathcal{E} = \mathcal{E}^*$ providing the minimal *size*. In this case, we talk about zone convergence or *practical stability* such that the effectiveness of the robust control strategies is associated with the *size* of the corresponding attractive ellipsoid set. The class of linear stabilizing feedback is given by the corresponding LMI (Linear Matrix Inequalities) or BMI (Bilinear Matrix Inequalities). If they are satisfied, then one may guarantee that all possible trajectories of the considered systems are bounded. Control systems with saturating actuators and state constraints allow the study of CQLF as a potential tool to handle more general nonlinearities. Then we can construct continuous feedback laws based on the gradient of the function or a given set of linear feedback laws. A CQLF is defined as a continuously differentiable function whose level set is the convex hull of a set of ellipsoids. In general, a piecewise quadratic function may not be continuously differentiable and the level sets may not be convex, see [12], [13] and references there in. To estimate the domain of attraction and construct the controllers to enlarge the domain of attraction, we introduce a new type of Lyapunov function in AEM stability analysis of control problem (3). According to [2], [12]–[15] the union of this set of invariant ellipsoids of the closed-loop system under a saturated feedback law is also an invariant set of the closed-loop system. The convex hull of this set of ellipsoids is a set potentially much larger than the union but may not be invariant.

The present contribution has the following structure, section II is devoted to reviewing the conventional theoretical results about *Linear* DAE, AEM, and CQLF. In section III we present problem formulation in terms of convex-type matrix constraints and the optimization problem to solve. Section IV deals with practical stability analysis of the transformed problem and presents the main analytical results. In section V we present a numerical implementation of an associated computational algorithm that is based on a matrix inequalities solution. Section VI summarizes the contribution. Finally, the reviewed references are presented.

II. Preliminaries

The necessary and sufficient conditions for the Linear DAE solution are presented. An AEM review illustrates the procedure to ensure the so-called *practical stability* and *robustness* of DAE solution (perturbation rejection) by using an admissible control strategy, numerically obtained. The AEM review is extracted from [5]. The CQLF section presents the set invariance properties for linear systems with input and state constraints for systems with a class of convex/concave nonlinearities.

A. Linear DAE

There are some important solvability conditions of Linear DAE reported in [7]–[10]. The matrix pair $(E(t), A(t))$ for all $t \in \mathbb{I}$ is called *regular*, where E, $A : \mathbb{I} \to \mathbb{R}^{n\times n}$, if the so-called *characteristic polynomial*, $p(\lambda, t) = \det(\lambda E(t) - A(t))$, is not the zero polynomial for all $t \in \mathbb{I}$. A matrix pair which is not regular is called singular. In the case of constant coefficients in descriptor system (3), every initial value problem with consistent initial condition is uniquely solvable. In the case of Linear DAE with variable coefficients, regularity of a matrix pair for every sufficiently smooth inhomogeneity $f(\cdot)$ does not guarantee unique solvability of the initial value problem. However, in the control context, it is possible to modify system properties of (3) using an adequate feedback and an adequate selection of matrix A, in particular to make non-regular systems regular or to change the index of the system.

B. Attractive Ellipsoid Methodology

AEM is restricted to a specific class of the Q-L functions $\phi(\cdot)$. Contributions [11], [16] deal with the case of a singular constant matrix E of (3). A vector function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be of the class $\mathcal{C}_\phi(A, \delta_1, \delta_2)$ of Q-L functions if there exist a matrix $A \in \mathbb{R}^{n\times n}$ and nonnegative constants δ_1 and δ_2 such that for every $x \in \mathbb{R}^n$, the following inequality holds: $\|\phi(x) - Ax\|^2_{Q_\phi} \leq \delta_1 + \delta_2 \|x\|^2_{Q_x}$. This implies that the growth rates of $\phi(x)$ as $\|x\| \to \infty$ are not faster than linear. Where Q_ϕ, $Q_x \in \mathbb{R}^{n\times n}$, $Q_\phi = Q_\phi^\intercal \geq 0$, $Q_x = Q_x^\intercal \geq 0$. By $\|\cdot\|_{Q_j}$, $j = \phi, x$ (where Q_j is a given suitable symmetric positive definite matrix) we denote a *weighted Euclidean norm*. We easily obtain an alternative description of the Q-L model (3):

$$\begin{aligned} E\dot{x} &= Ax + Bu + \sigma(x(t), \phi(x)) \\ y &= Cx + \kappa(t), \end{aligned} \tag{4}$$

where $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $\sigma(x, \phi) := \phi(x) - Ax$. The condition of Q-L can be considered not only as a kind of a *linearization* procedure applied to a known function $\phi(\cdot)$ but also as a priori estimate for of the perturbation associated with a given system (4). According to [5], a set Ω in the state space is said to be *positively invariant* if every trajectory is initiated in this set remains inside the set at all future time. Also, Ω is Lyapunov *globally asymptotically attractive* for the same system where $x_0 \notin \Omega$, if every solution of the problem tends to Ω, as t tends to infinity. If the closed-loop system (4) with $u(x) = Kx$ and $K \in\to \mathbb{R}^{m\times n}$, is a globally asymptotically attractive invariant set of a system (4), we said that $\mathcal{E}$ is an attractive ellipsoid. The attractive property doesn't imply the Lyapunov asymptotic stability of the invariant set. Lyapunov function or CQLF method provides useful tools for stability, robustness analysis, and control design for nonlinear control systems such that $\mathcal{E}$ is a globally asymptotically stable positively invariant set of minimal size for the system (4). Function $V : \mathbb{R}^n \to \mathbb{R}$ is proper if it is continuously differentiable in $\mathbb{R}^n$; it is positive finite ($V(x) > 0$ for $x \neq 0$, $V(0) = 0$); and it is radially unbounded ($\|x\| \to +\infty$ implies $V(x) \to +\infty$). AEM in [5] makes it possible to specify constructively an attractive invariant set for a class of dynamic processes with Q-L right-hand sides. The corresponding robust control design schemes become LMI or BMI constraints. Minimizing problem evidently includes some natural additional restrictions for the *free* parameters, namely for P and for the gain matrix K from control law $u = Kx$. It is important to note that the AEM main theorem (ODE-case) uses the Lie derivative of function $V(\cdot)$ however, in the DAE case, we will use the corresponding descriptor derivative reported in [17]. The main goal in this contribution is to extend these results to the system (3) using a new type of Lyapunov functions: CQLF.

C. Composite Quadratic Lyapunov Functions

According with [2], [12]–[15]: For two integers k_1, k_1, where $k_1 < k_2$, also we denote $I[k_1, k_2] := \{k_1, k_1 + 1, ..., k_2\}$. With a positive definite matrix $P > 0$, $P \in \mathbb{R}^{n\times n}$ a quadratic function can be defined as $V(x) := x^\intercal P x$. For a positive number ρ, a level set of $V(\cdot)$, denoted $L_V(\rho)$ is $L_V(x) := \{x \in \mathbb{R}^n : V(x) \leq \rho\} = \mathcal{E}(P, \rho)$. We are interested in a function determined by a set of positive definite matrices $P_1, P_2, ..., P_N \in \mathbb{R}^{n\times n}$. Let $Q_j = P_j^{-1}$, $j \in I[1, N]$. For a vector $\gamma \in \mathbb{R}^N$, define $Q(\gamma) := \sum_{j=1}^N \gamma_j Q_j$, $P(\gamma) := Q^{-1}(\gamma)$. Let $\Gamma := \{\gamma \in \mathbb{R}^N : \sum_{j=1}^N \gamma_j = 1, \gamma_j \geq 0, j \in I[1, N]\}$. It is easy to see that $Q(\gamma)$, $P(\gamma) > 0$ for all $\gamma \in \Gamma$ and these two matrix functions are analytic in $\gamma \in \Gamma$, i.e., each function is locally given by a convergent power series.

The composite quadratic Lyapunov function is defined as $V_c(x) := \min_{\gamma\in\Gamma} x^\intercal P(\gamma) x$. Clearly $V_c(\cdot)$ is a positive definite function. For $\rho > 0$, the level set of $V_c(\cdot)$ is $L_{V_c}(\rho) := \{x \in \mathbb{R}^n : V_c(x) \leq \rho\}$. A very useful property of this composite quadratic Lyapunov function is that its level set is the convex hull of the level sets of $x_j^\intercal P_j x_j$, the ellipsoids $\mathcal{E}(P_j, \rho)$, $j \in [1, N]$. Another nice property of $V_c(\cdot)$ is that it is continuously differentiable with the partial derivative $\frac{\partial V_c}{\partial x} = 2P(\gamma^*)x$. Where $\gamma^*(x)$ is the optimal value of γ for $V_c(x)$.

Note here that it is possible to generate $V_{c_1}(x) := \max_{\gamma\in\Gamma} x^\intercal \left(\sum_{j=1}^N \gamma_j Q_j\right)^{-1} x$ or $V_{c_2}(x) := \max_{\gamma\in\Gamma} x^\intercal \left(\sum_{j=1}^N \gamma_j P_j\right) x$ as another CQLF. As to $V_{c_1}(\cdot)$, we note that for a fixed $x^\intercal P(\gamma)x$ is a convex function of γ. Hence, its maximum is attained at the vertices of Γ . It follows that $V_{c_1}(\cdot) = V_{c_2}(\cdot)$. The computation of these functions is easy and straightforward, but they are not well behaved as compared with $V_c(\cdot)$. It can be verified that the level set of $V_{c_1}(\cdot)$ and $V_{c_2}(\cdot)$ is the intersection of the ellipsoids $\mathcal{E}(P_j, \rho)$, $j \in I[1, N]$. The level set has nonsmooth surfaces and the functions $V_{c_1}(\cdot)$ and $V_{c_2}(\cdot)$ have nondifferentiable points. We see that the optimal value of γ for $V_c(x)$ is $\gamma^*(x)$ such that $V_c(x) = x^\intercal P(\gamma^*)x$. In some situations, the optimal value of γ is not unique. For example, this may happen if some Q_j can be expressed as the convex combination of other matrices in the set.

The convex combination of $x_1, x_2, ..., x_k \in \mathbb{R}^n$ is an element of the form $\sum_{j=1}^k \gamma_j x_j$, $\gamma \in \Gamma_k$. A set S is convex if and only if it contains every convex combination of its elements. The convex hull of a set S, denoted $co\{S\}$, is the set of all the convex combinations of the elements in S. If $S = \cup_{j=1}^k C_j$ and each C_j is convex, then $co\{S\} := \{\sum_{j=1}^k \gamma_j x_j : \gamma \in \Gamma_k, x_j \in C_j, j \in [1, k]\}$.

III. Problem Formulation

Consider the *Initial Valued Problem* (IVP) described by following nonlinear DAE with variable coefficients:

$$E(t)\dot{z}(t) = \phi(z(t)) + B(t)u(t) + \eta(t) \tag{5}$$

$$y(t) = C(t)z(t) + \xi(t), \quad z(0) = z_0, \ t \in \mathbb{R}^+, \tag{6}$$

where $E : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $C : \mathbb{R}_+ \to \mathbb{R}^{q\times n}$ are continuous matrix functions, $\det(E) \neq 0$ singular for all $t \in \mathbb{R}_+$, $B(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$. $z(t) \in \mathbb{R}^n$, $u(z) \in \mathbb{R}^m$, denote n-dimensional descriptor variable and m-dimensional control input vector, respectively. $y \in \mathbb{R}^q$ is the output vector.

System (5)-(6) are under particular hypothesis:

- $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable nonlinear function, Q-L whose derivative is simply bounded.

- $\eta(t) \in \mathbb{R}^n$, $\xi(t) \in \mathbb{R}^q$, $\eta \in \mathcal{C}(\mathbb{I}, \mathbb{R}^n)$, $\xi \in \mathcal{C}(\mathbb{I}, \mathbb{R}^q)$, vector functions for some interval $\mathbb{I} \in \mathbb{R}_+$. They are not only unknown and deterministic perturbation terms but also bounded and nonlinear. That is $\eta(\cdot)$, $\xi(\cdot)$ are functions such that $||\eta(t)||_{Q_\eta} + ||\xi(t)||_{Q_\xi} \leq 1$, $\forall t \in \mathbb{R}^+$. Where Q_η, $Q_\xi \in \mathbb{R}^{n\times n}$, $Q_\eta = Q_\eta^\intercal > 0$, $Q_\xi = Q_\xi^\intercal > 0$.
- $u(z) = K(t)z$ is the descriptor variable feedback input, where $K(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{m\times n}$ is a time-variant gain matrix.

Control aim is to find a gain matrix function, $K(t)$, respect to the descriptor variable feedback system (5)-(6), which guarantee practical stability of solution trajectories, $z(t)$, for all consistent initial condition. A general solution scheme of the solution idea for the initial valued problem, (5)-(6), is presented below. The methodology is based on the solution presented in [11].

Linear DAE is obtained from (5) performing the following algebraic calculations.

i) Transform the nonlinear EDA into a linear EDA using a linearization-like technique of AEM. **If it is not possible just consider as the problem formulation a control problem described by the following Linear Time Variant System**

$$E(t)\dot{z}(t) = A(t)z(t) + B(t)u(t) + \sigma(z(t), \eta(t)), \quad (7)$$

where $\sigma(z(t), \eta(t)) := \phi(z(t)) - A(t)z(t) + \eta(t)$ is a new perturbation term. **This term is not considered in the case of LTVS problem formulation.**

ii) Obtain the feedback system using $u(z) = K(t)z$.

$$E(t)\dot{z}(t) = (A(t)+B(t)K(t))z(t)+\sigma(z(t), \eta(t)), \quad (8)$$

As we mentioned at introduction and section 2.1, regularity property of pair matrix $(E(t), A(t))$, for all $t \in \mathbb{R}_+$ is a necessary and sufficient condition to ensure the solution of the differential-algebraic equation for every consistent initial and for every sufficiently smooth inhomogeneity $\sigma(x(t), \eta(t))$. See [7], Theorem 2.7 of page 16. However, if pair matrix $(E(t), A(t))$ for system 8 is not regular for all $t \in \mathbb{R}_+$, in terms of control, it is possible to modify system properties using proportional state or proportional output feedback. Thus, this feedback can be used to modify the system properties, in particular, to make non-regular systems regular or to change the index of the system. Q-L condition for system 8 ensure the existence of matrix $A(t)$, and therefore its linear form. In the numerical procedure summarized in section III together, the arbitrariness property of A matrix and feedback gain matrix selection will become an important solution strategy. See [7], Theorem 2.56 in page 51.

iii) Write a new IVP using the semi-explicit DAE form and define a general hypothesis on this.

$$\begin{aligned} E(t)\dot{z}(t) &= A(t)z(t) + B(t)u(t) + \sigma(z(t), \eta(t)) \quad (9)\\ y(t) &= C(t)z(t) + \xi(t), \quad z(0) = z_0, \ t \in \mathbb{R}^+ (10)\end{aligned}$$

under hypothesis:

- $\sigma(\cdot, \cdot) \in \mathcal{C}(\mathbb{I}, \mathbb{R})$ is an unknown deterministic and bounded perturbation function such that $||\sigma(x(t), \eta(t))||_{Q_\sigma} \leq ||\phi(x(t)) - A(t)z(t)||_{Q_f} + ||\eta(t)||_{Q_\eta} \leq (1+\delta) + h||z(t)||_{Q_z}$. Where $\mathbb{I}$ is an interval in $\mathbb{R}^+$. Where Q_σ, Q_f, $Q_z \in \mathbb{R}^{n\times n}$, $Q_f = Q_f^\intercal > 0$, $Q_\sigma = Q_\sigma^\intercal > 0$, $Q_x = Q_z^\intercal > 0$. **If it is not possible just consider as the problem formulation a control problem described by the following Linear Time Variant System**
- $u(z) = K(t)z(t)$ is the linear descriptor variable feedback control.

Control aim now is to find a constant gain matrix, K^*, respect to the descriptor variable feedback system (9)-(10), which guarantee practical stability of solution trajectories, $z(t)$, for all consistent initial condition. Aforementioned assumptions are all compatible with assumptions of (5)-(6). Notice that matrix $A(t)$ selection is equivalent to matrix J selection for regular pair matrix $(E(t), A(t))$.

A. On the Systems Characterized by the Convex-Type Matrix Constraints

In some cases it is possible to consider a reference pair $(u_{ref}(\cdot), z_{ref}(\cdot))$ such that the resulting parameter matrices remain bounded inside a convex hull, namely, $E(t) \in co\{E_1, E_2, \dots, E_N\}$, $A(t) \in co\{A_1, A_2, \dots, A_N\}$ and $B(t) \in co\{B_1, B_2, \dots, B_N\}$. The above convex restrictions constitute an adequate modeling framework in the case of a system

$$E(t)\dot{z} = A(t)z + B(t)u + \sigma(z, \eta) \quad (11)$$

with $E = \sum_{i=1}^N \lambda_i(t)E_i$, $A = \sum_{i=1}^N \lambda_i(t)A_i$, $B = \sum_{i=1}^N \lambda_i(t)B_i$. Here $\lambda_1(t)$ are some positive semi-definite continuous functions, $\sum_{i=1}^N \lambda_i(t) = 1$ for all $t \geq 0$. Taking into consideration the above framework it is possible to propose a time varying F (generated by a combination of some convex functions) such that the solution of the resulting closed-loop system converges to a minimal size set generated by the convex hull of some ellipsoids.

B. Optimization Problem Formulation

The optimization problem can be formulated as follows:

Problem III.1. *Find constant gain matrix, K^*, respect to the feedback system (9)-(10), which guarantee practical stability of its solution trajectories, $z(t)$, converging to the attractive ellipsoid, defined by $\mathcal{E}(P) := \{z \in \mathbb{R}^n \mid z^\intercal Pz \leq 1, \quad P \in \mathbb{R}^{n\times n}, \quad P > 0, P^\intercal = P\}$, of minimum area; is equivalent to solve next optimization problem:*

$$\begin{aligned} &\text{minimize } \operatorname{tr}[P] \\ &\text{subject to } \ P > 0, \ P = P^\intercal, \ K^* \in \Upsilon, \ L \in \Omega, \end{aligned} \quad (12)$$

where $\Upsilon \subset \mathbb{R}^{m\times n}$ and $\Omega \subset \mathbb{R}^{n\times q}$ are the admissible control and observer sets which ensure invariance of attractive *ellipsoid $\mathcal{E}(P)$, where; provided that $t \to \infty$ and all consistent initial condition x_0.*

Analytical characterization of sets, Υ, and Ω, allow developing a feasible and easy to implement numerical-algorithm which provides a computational approach in solving optimization Problem III.1. This characterization will be developed considering Lyapunov-like stability analysis approach and the use of the bilinear matrix inequalities (BMI) framework.

IV. Practical Stability

The main result of this contribution, in the form of the following theorem, allows ensuring the practical stability of solution trajectories of the DAE system (9)-(10), under feasible initial conditions. Similar to the Theorem 2 in [18], which requires the use of a *CQLF*, following theorem requires the use of a *storage function* definition, $V : \mathbb{R}^n \to \mathbb{R}^+$, valuated along solution trajectories $z(t)$.

In this section, we apply the theoretical results obtained above and propose an implementable computational algorithm for the robust control design for the system (5). It also includes two practically oriented numerical examples. The main assumption of the linearization-like process is the so-called Q-L condition. In the case of an affine non-stationary system (5) this restrictive condition can formally be written as follows: $\|\phi(t,z) + B(t)u - A(t)z\|^2_{Q_\sigma} \le \delta_1 + \delta_2\|z\|^2_{Q_z}$, where $\delta_1,\ \delta_2 > 0$. The previous condition has no sense for the basic non-stationary system (5). The right-hand side of this inequality does not depend on the variable t. Note that this absence of the "dynamics" in inequality conditions makes it impossible to apply the conventional AEM even in the case of systems with a simple non-stationary structure (see academic example in section V). The most important result in that direction was given in [12], [13]. It has been proven that it is possible to generate a continuous feedback control law such that a convex hull of ellipsoids constitutes an invariant and attractive set, for a time-invariant system. We now extend this result to the time-varying case when the system matrices are given as in (11).

Theorem IV.1. *Assume there exists some positive definite symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, some matrices $R_i \in \mathbb{R}^{m\times n}$, $i = 2,3$ and some positive semi-definite continuous functions $\gamma_i(t)$, $\sum_{j=1}^N \gamma_j(t) = 1$, $\gamma_D := sup\{\|\dot\gamma_j(t)\| : t \ge 0, j = 1,\dots,N\}$. Let $\alpha,\delta > 0$, $\beta := 1+\delta > 0$ and $V(z) := z^\intercal P z$, $P = P^\intercal$, $P > 0$, $P \in \mathbb{R}^{n\times n}$. Derivative of storage function, $DV(z(t))$, is calculated using the so-called* descriptor methodology, *developed in [17]. Then $V(z(t))$ satisfy the condition $V(t) \le \frac{\beta}{\alpha} + (V(0) - \frac{\beta}{\alpha})\exp(-\alpha t)\ \forall t \ge 0$ if the inequality:*

$$DV(z(t)) + \alpha V(z(t)) - \beta \le W^\intercal(t) Z(R_i, K, \alpha) W(t) < 0,$$

holds. Where $DV(z(t)) := \dot V(z(t)) + 2\langle[Pz + R_2\dot z + R_3\sigma], [-E(t)\dot z(t) + A(t)z + B(t)u(t) + \sigma(z,t)]\rangle$, with $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$, $W(t) := (z^\intercal(t), \dot z^\intercal(t), \sigma^\intercal(t))^\intercal$ is an extended descriptor variable, $Z(R_i, K, \alpha)$ is a matrix function such that $Z < 0$, for $i = 2,3$. Also the set $\mathcal{E}(P(t)) := \{z(t) \in \mathbb{R}^n : z(t)^\intercal P(t) z(t) \le 1, \forall t \ge 0, \gamma \in \Gamma\}$, $P(t) := \sum_{i=j}^N \gamma_j(t) P_j$, is an invariant set for system (11) with the control input $u(z) = K(t)z(t)$, and

$$K^* := \left(\sum_{j=1}^N \gamma_j(t) Y_i\right)\left(\sum_{j=1}^N \gamma_j(t) P_j\right)^{-1}. \tag{13}$$

Proof. :

A sketch of the Theorem proof include the so-called descriptor method (see [17]) which allows to estimate derivative of $V(z) = z^\intercal(t)P(t)z(t)$ using:

$$\begin{aligned} DV(z(t)) = \dot V(z(t)) + 2\langle[R_1 z(t) + R_2\dot z(t) + R_3\sigma], [-E(t)\dot z(t) \\ + A(t)z(t) + B(t)u(t) + \sigma]\rangle, \end{aligned} \tag{14}$$

where unknown matrices $R_i \in \mathbb{R}^{n\times n}$, $i = 1,2,3$. Let $W(t) = (z^\intercal(t), \dot z^\intercal(t), \sigma^\intercal(t))^\intercal$, we write z instead of $z(t)$ and σ instead of $\sigma(z,\eta)$. $<\cdot,\cdot>$ represent the inner product in the n-dimensional space $\mathbb{R}^n$. Note also that first three terms constitute the temporal derivative of energetic function $V(z)$. According to [17] the rest of terms are all zero effective terms and includes the dynamics of the perturbation term in the DAE. Considering the so-called adjoint form of DAE, defined by $\mathcal{F}(t) := \dot V(z) + \alpha V(z) - \beta$, the additional zero term $\pm\frac{\beta}{M}\sigma^\intercal\sigma$ and the extended vector $W = (\dot z^\intercal\ \ z^\intercal\ \ \sigma^\intercal)^\intercal$ we calculate adjoint form in its general matrix form $\mathcal{F}(z) = W^\intercal \bar Z(t) W(t) + \frac{\beta}{M}\sigma^\intercal\sigma - \beta$, where

$$\bar Z = \begin{pmatrix} \begin{matrix} R_1^\intercal(A+BK) + (A^\intercal + \\ K^\intercal B^\intercal)R_1 + \alpha P + \dot P \end{matrix} & \begin{matrix} P - R_1^\intercal E + \\ (A^\intercal + K^\intercal B^\intercal)R_2 \end{matrix} & R_1^\intercal + (A^\intercal + K^\intercal B^\intercal)R_3 \\ \begin{matrix} P^\intercal - E^\intercal R_1 + \\ R_2^\intercal(A+BK) \end{matrix} & -R_2^\intercal E - E^\intercal R_2 & R_2^\intercal - E^\intercal R_3 \\ R_1 + R_3^\intercal(A+BK) & R_2 - R_3^\intercal E & R_3^\intercal + R_3 - \frac{\beta}{M} I_n \end{pmatrix}, \tag{15}$$

we defined $Q := P^{-1}$, $Q_i := R_i^{-1}$ for $i = 2,3$, and $Y(t) = K(t)Q(t)$. We note also that $\dot Q = -Q\dot P Q$, to pre-multiply $\bar Z$ by diag$\{R_1^{-\intercal}, R_2^{-\intercal}, R_3^{-\intercal}\}$ and post-multiply $\bar Z$ by diag$\{R_1^{-1}, R_2^{-1}, R_3^{-1}\}$, where $R_i^{-\intercal} := (R_i^{-1})^\intercal$, also considering $R_1 = P$ the symmetric positive definite matrix which describes energetic function and the ellipsoidal region. Also it is possible to find an upper bound for $-\dot P(t)$ as follows $-\dot Q = \sum_{j=1}^N \dot\gamma Q_j^{-1} \le \bar\gamma_D \sum_{j=1}^N Q_j^{-1}$, using this and assumption (2), we obtain the upper bound for $\mathcal{F}(z)$: $\mathcal{F}(z) \le z^\intercal(t) Z(t) z(t)$, where Z is equal to

$$\begin{pmatrix} \begin{matrix} AQ + BY + Q^\intercal A^\intercal + \\ Y^\intercal B^\intercal + \alpha Q + \bar\gamma_D \sum_{j=1}^N Q_j^{-1} \end{matrix} & \begin{matrix} Q_2 - EQ_2 + \\ Q^\intercal A^\intercal + Y^\intercal B^\intercal \end{matrix} & Q_3 + Q^\intercal A^\intercal + Y^\intercal B^\intercal \\ \begin{matrix} Q_2^\intercal - Q_2^\intercal E^\intercal + \\ AQ + BY \end{matrix} & -EQ_2 - Q_2^\intercal E^\intercal & Q_3 - Q_2^\intercal E^\intercal \\ Q_3^\intercal + AQ + BY & Q_3^\intercal - EQ_2 & Q_3 + Q_3^\intercal - \frac{\beta}{M} I_n \end{pmatrix}, \tag{16}$$

If $W^\intercal(t)Z(t)W(t) < 0$ is true then $Z < 0$ and $z^\intercal(t)Z(t)z(t) \le \alpha(1 - z^\intercal(t)P(t)z(t))$ which implies the desired result: $W^\intercal(t)Z(t)W(t) < 0$ if $z^\intercal(t)P(t)z(t) > 1$. The proof is completed.

Remark IV.1. *To ensure the differentiable property of $V(z)$ it is possible to select*

$$V(t) = \min_{\gamma} z^{\intercal}(t) \left(\sum_{j=1}^{N} \gamma_j(t) P^{-1} \right)^{-1} z(t) \qquad (17)$$

in [12], [13] it has been shown that the level set $V(z) = 1$ for this function is the convex hull of the ellipsoidal sets characterized by the matrix P_i, $i = 1, \ldots, N$. Additionally, it is possible that for some specific systems with set of matrices A_i and B_i there exists a constant solution for $\gamma_i(t) = \gamma_i$ for all $i = 1, \ldots, N$, for example in the case of quadratic stable polytopic systems.

Theorem IV.1 allows to characterize in an easily implementable way the set Υ of optimal control Problem III.1 trough solution of LMI condition: $Z(P, R_i, K, \alpha) \leq 0$, $i = 2, 3$, under a fixed scalar parameter α. Optimal control Problem III.1 can be rewriting in the following form:

Problem IV.1. *Find gain matrix, $K(t)$, respect to the feedback system (9)-(10), which guarantee practical stability of its solution trajectories, $z(t)$, converging to the attractive ellipsoid convex hull, defined by intersection of ellipsoids $\mathcal{E}(P) := \{z \in \mathbb{R}^n \mid z^{\intercal} P z \leq 1, \;\; P \in \mathbb{R}^{n \times n}, \;\; P > 0, P^{\intercal} = P\}$, of minimum area; is equivalent to solve next optimization problem:*

$$\begin{aligned} &\text{minimize} \;\; \mathrm{tr}[P] \\ &\text{subject to} \;\; P > 0, \;\; P = P^{\intercal}, \\ &Z(P, R_i, K, \alpha) < 0, i = 2, 3 \end{aligned} \qquad (18)$$

Once LMI is solved, following theorem will set the equivalence relation between solutions of transformed system (9)-(10) and system described by (5)-(6). The solution vector $(P^{subopt}, R_i^{subopt}, K^{subopt}, \alpha)$, $i = 2, 3$, of convex Problem IV.1 under a fixed scalar parameter α, defines a suboptimal solution due to α-values quest is limited to a finite set of $\mathbb{R}_+$.

Theorem IV.2. *If transformed optimization convex problem IV.1, has a suboptimal solution $(P^{subopt}, R_i^{subopt}, K^{subopt})$, $i = 2, 3$ in the sense of minimum trace of ellipsoid convex hull for a fixed scalar parameter α, then 4-tuple $(P^{subopt}, R_i^{subopt}, K^{subopt}, \alpha)$, $i = 2, 3$ is also a suboptimal solution to IVP described by (5)-(6).*

Proof: A detailed proof of the theorem is presented in [16]. □

Transformed optimal Problem IV.1 set a theoretical approach to deal with the numerical treatment of IVP described by (5)-(6), just considering computational simulation of an LMI.

V. Numerical example

Here a numerical academic example is presented. It is an academic example that exposes the basic idea of optimization Problem III.1 solution. In this case, the well known PENOPT toolbox from MatLab software is used to solve LMI, which includes the YALMIP toolbox and PENBMI solver.

A. Academic example

Optimization Problem III.1, associated with system (9)-(10) has following values: $E = \mathrm{diag}\{1, 1, 1, 0\}$, $C = (0, 1, 0, 0)$, with $A(t) = \begin{pmatrix} 1 + 2\cos(t) & \frac{1}{2}(3\sin(t) - 1) & \frac{1}{2}(1 - 3\cos(t)) & -\frac{1}{2}(1 - 3\sin(t)) \\ -\frac{1}{2}(1 + 3\cos(t)) & 1 + 3\sin(t) & -\frac{1}{2}(3\cos(t) - 1) & 1 \\ -1 & \frac{1}{2}(-3 + \cos(t)) & 1 + 2\cos(t) & 1 \\ \frac{1}{2}(3\cos(t) - 1) & \frac{1}{2}(3\sin(t) - 5) & \frac{1}{2}\sin(t) - \frac{1}{2}) & \frac{1}{2}(3 - 5\cos(t)) \end{pmatrix}$, and $B(t) = \begin{pmatrix} \frac{1}{2}(1 - 3\sin(t)) & \frac{1}{2}(1 + \cos(t)) \\ 1 + \cos(t) & \frac{1}{2}(1 + 3\cos(t)) \\ -\frac{1}{2}(\sin(t) + 1) & \frac{1}{2}(1 - 3\sin(t)) \\ \frac{3}{2}(1 + \cos(t)) & -\frac{1}{2}(\sin(t) + 1) \end{pmatrix}$. Where A_1, A_2, B_1 and B_2 are the corresponding lower and upper bounds of matrices $A(t)$ and $B(t)$, respectively:

$$A_1 = \begin{pmatrix} -1 & -2 & -1 & -2 \\ -2 & -2 & -1 & 1 \\ -1 & -3 & -1 & 1 \\ -2 & -4 & -1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 4 & 2 & 1 \\ -1 & -2 & 3 & 1 \\ 1 & -1 & 0 & 4 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 1 \\ 2 & 2 \\ 0 & 2 \\ 3 & 0 \end{pmatrix},$$

and the functions $\lambda_1(t) = 0.5 + 0.5\sin(10t)$, $\lambda_2(t) = 0.5 + 0.5\sin(10t)$. The obtained matrices are

$$P_1^{-1} = \begin{pmatrix} 0.1278 & -0.0261 & 0.0448 & -0.0301 \\ -0.0261 & 0.2081 & 0.0033 & 0.0216 \\ 0.0448 & 0.0033 & 0.1387 & 0.0174 \\ -0.0301 & 0.0216 & 0.0174 & 0.1949 \end{pmatrix},$$
$$P_2^{-1} = \begin{pmatrix} 0.6096 & -0.1967 & -0.0134 & -0.4423 \\ -0.1967 & 0.9009 & -0.0067 & -0.2227 \\ -0.0134 & -0.0067 & 0.9995 & -0.0152 \\ -0.4423 & -0.2227 & -0.0152 & 0.4993 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 0.055 & 0.087 & 0.034 & 0.056 \end{pmatrix}, Y_2 = \begin{pmatrix} 0.075 & 0.1 & 0.022 & 0.069 \end{pmatrix},$$

The numerical results obtained in this example illustrate the effectiveness and implementability of the proposed robust control design. Numerical solution of optimization Problem III.1 satisfy all conditions of Theorem IV.1. Main condition must be satisfied is BMI: $Z \leq 0$. Then quasi-optimal values of K, and P were used to proved practical stability of nonlinear initial system described by (5)-(6). MatLab, through PENOPT, solve BMI in an optimal way and found values of

$$K_s^{qopt} = \begin{pmatrix} -406.077 & -519.126 & -521.161 & 661.379 \\ -406.082 & -519.107 & -521.168 & 661.351 \end{pmatrix},$$

and $\mathrm{eig}(P^{qopt}) \in [0.999\,94, 33\,469.4712] \subset R_+$. The above mentioned values were used in numerical simulation of IVP described by (5)-(6), where $\phi(z) = (\phi_1(z) \;\; \phi_2(z) \;\; \phi_z(x) \;\; \phi_4(z))^{\intercal}$, $\phi_1 = -z_1 - |z_2| + \sin(z_2) + 0.1z_3$, $\phi_2 = -0.1z_1 - z_2 + \cos(z_3)$, $\phi_3 = -|z_1| - 0.1\cos(z_2) - z_3$, $\phi_4 = 0.1z_2$ and $\eta = (0.02sin(t), -0.05cos(t), -0.05cos(t), 0.03cos(t))^{\intercal}$, $\xi = 0.1sin(t)$ with initial condition $z_0 = (100, -500, 200, 100)^{\intercal}$. Results reported here include time evolution of descriptor variable components, $z(t)$, in Fig. 1; phase portrait performed in z_1, z_2 and z_3 space which Ellipsoid form is also included, see Fig. 2.

VI. Conclusions

Considering the use of the so-called Composite Quadratic Lyapunov Functions (CQLF) based on descriptor techniques from the stability theory of continuous systems, this paper designs a linear feedback control to deal with bounded uncertainties in a class of nonlinear dynamical system described

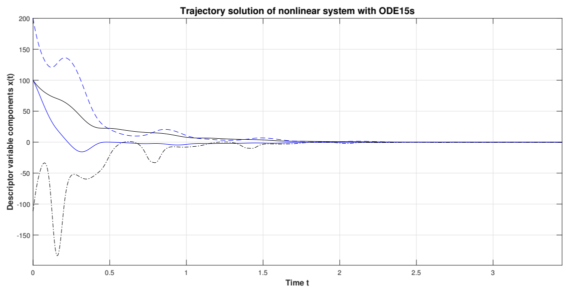

Fig. 1. Descriptor variable components z(t) of nonlinear system.

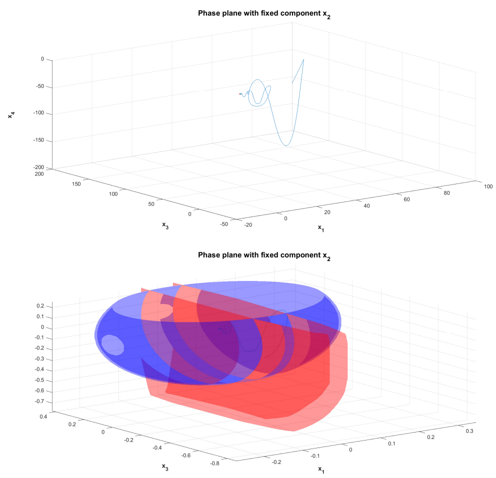

Fig. 2. (Upper) a) Phase plane of trajectory solution emerging from IC, where component z_2 is a fixed value. (Down) b) Zoom around origin on phase plane and convex hull Ellipsoidal region.

by semi-explicit differential-algebraic equations (DAE) by application of an extended version of the conventional Attractive Invariant Ellipsoid Method (AIEM or AEM). To find a solution to the optimization control problem a parameter γ was used to the construction of continuous feedback law $u(z) = K(t)z$ and the construction of the convex hull of a group of invariant ellipsoids. Regularity of a matrix pair $(E(t), A(t))$ was ensured for all $t \in \mathbb{I}$ with election of matrix $A(t)$. The suboptimal solution, obtained numerically in terms of an LMI, constrained all possible trajectories of the system as bounded inside a convex hull of minimal size. An academic example was performed to illustrate this zone of convergence or *practical stability* is an effective robust control strategy.

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