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Observer-Based Leader-Follower Consensus Tracking with Fixed-time Convergence

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***Abstract*—This paper addresses the problem of consensus tracking with fixed-time convergence, for leader-follower multi-agent systems with double-integrator dynamics, where only a subset of followers has access to the state of the leader. The control scheme is divided into three steps. The first one is dedicated to the estimation of the leader's velocity by the subset of agents with access to leader's position. Then, the leader state is estimated by each follower in a distributed way and in a fixed-time. Finally, based on the estimate of the leader state, each follower computes its control law to track the leader in a fixed-time. The control strategy proposed in this paper ensures a fixed-time convergence where the desired upper bound of the settling-time of the tracking error is set a priory by the user with one parameter. Thus, allowing the application for scenarios under time constraints. Some numerical examples show the effectiveness of the proposed consensus protocols.**

***Index Terms*—Leader-follower consensus, Fixed-time convergence, Multi-agent systems, Fixed-time observer.**

I. Introduction

In the last years, the problems of coordination and control of Multi-Agent System (MAS) have been widely studied (see for instance [1]–[4]), due mainly to the ability of a MAS to face complex tasks that a single agent is not able to handle. Distributed control approaches applied to a MAS require a communication network allowing to share information with a subset of agents (neighbors). In this context, several interesting problems and applications have been investigated in the literature, for instance, synchronization of complex networks [5], consensus [6] and formation control of multiple agents [7]. Among all the mentioned problems, an interesting one is the leader-follower consensus problem where a set of agents, through local interaction, converges to the state of a leader, even though the leader may not be accessible for all agents.

The consensus problem consists in reaching a common agreement state by exchanging only local information [6], [8]. Linear average consensus protocols with asymptotic convergence were proposed in [6], [8]. It has been demonstrated that the second smallest eigenvalue of the Laplacian graph (i.e. the algebraic connectivity) determines the convergence rate of the MAS. Furthermore, the problem of tracking a reference by a MAS (i.e. leader-follower consensus problem) has been investigated where the common agreement to reach is the state of a reference imposed by a leader which evolves independently of the MAS [9], [10]. However, in these works the convergence is only asymptotic.

To improve the convergence rate of a MAS, finite-time stability in consensus problems has been studied in [11]. However, the settling time is an unbounded function of the initial conditions of the system. Therefore, the concept of fixed-time stability has been introduced and applied to systems with time constraints [12]. In this case, the settling time is bounded by a constant which is independent of the initial conditions of the system. In the literature, there are several contributions on algorithms with fixed-time convergence, such as state observers [13], multi-agent coordination [14], [15], online differentiation algorithms [16], etc. In this paper, we address the leader-follower consensus problem of a MAS when the leader's velocity is not available. Then, each agent of the MAS tracks the trajectory of the leader using local available information, and we provide the necessary conditions to achieve the convergence in a fixed-time.

A Lyapunov differential inequality for an autonomous system to exhibit fixed-time stability was presented in [12], [17]. Based on this methodology, the works [18], [19] address the consensus tracking problem for high-order MAS, i.e., the MAS follows a trajectory imposed by the leader. However, in all these works, the *UBST* is not set as a design parameter and cannot be easily tuned. Therefore, the design of fixed-time leader-follower consensus algorithms where the *UBST* is set explicitly as a parameter of the system, is of a great interest.

In the leader-follower consensus problem, the state of the leader must be known by a subset of agents to track the trajectory of the leader. However, if the state of the leader is not available for the followers, the tracking cannot be done correctly. In [20], [21], control schemes where only part of the leader state is available (position) are presented, and the rest of the state is estimated by a subset of followers. Then, the leader-follower consensus is achieved with partial information of the leader. Nevertheless, the convergence to the leader trajectory is asymptotic.

In this paper, we present a methodology to achieve leader-follower consensus with fixed-time convergence. It consists in three steps. The first one estimates the leader's velocity by the subset of agents that has direct access to leader's position using a differentiator with fixed-time convergence. Then, the second step estimates the leader states (position and velocity) using a fixed-time observer that only requires the information of the neighbors. Then, the third step computes the control law to drive the followers to the estimated states in a fixed-time. The convergence for the control strategy is in fixed-time, where the *UBST* is established a priory by the user.

This work is structured as follows. Section II recalls some definitions and results from graph theory, and preliminaries on finite-time and fixed-time convergence are presented. In Section III, the problem of consensus tracking with fixed-time convergence is formulated. Section IV introduces the solution proposed to solve the consensus tracking problem when the leader's velocity is not available. Numerical results using the methodology proposed are presented in Section V. Finally, the conclusions are presented in Section VI.

II. Preliminaries

A. Graph theory

The following notation and preliminaries on graph theory are taken mainly from [22].

A graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}(\mathcal{G})$ and an edge set $\mathcal{E}(\mathcal{G})$ where an edge is an unordered pair of distinct vertices of $\mathcal{G}$. Writing ij denotes an edge, and $j \sim i$ denotes that the vertex i and vertex j are adjacent or neighbors, i.e., there exists an edge ij. The set of neighbors of vertex i in the graph $\mathcal{G}$ is represented by $\mathcal{N}_i(\mathcal{G}) = \{j : ji \in \mathcal{E}(\mathcal{G})\}$.

A weighted graph includes a weight function $W : \mathcal{E}(\mathcal{G}) \to \mathbb{R}_+$ on its edges. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ of a graph with N vertices is a square matrix where a_{ij} corresponds to the weight of the edge ij, when i is not adjacent to j then $a_{ij} = 0$. Through this work it is assumed that $a_{ij} = a_{ji}$, i.e. only *undirected* and *balanced graphs* are considered where $\sum_j a_{ij} = \sum_j a_{ji}$. The Laplacian matrix of $\mathcal{G}$ is $L(\mathcal{G}) = D - A$ where $D = \text{diag}(d_1, \cdots, d_N)$ with $d_i = \sum_{j=1}^{N} a_{ij}$.

A path from i to j in a graph is a sequence of distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of the graph $\mathcal{G}$, then $\mathcal{G}$ is said to be connected, otherwise it is said to be disconnected. If the graph $\mathcal{G}$ is connected, then the eigenvalue $\lambda_1(L(\mathcal{G})) = 0$ has algebraic multiplicity one with eigenvector $\mathbf{1} = [1 \ \cdots \ 1]^T$, i.e. $\ker L(\mathcal{G}) = \{x : x_1 = \ldots = x_n\}$.

Let $\hat{\mathcal{G}}$ be a weighted graph among all the agents (i.e. the leader and the followers). Then, the communication matrix between all the agents is represented by $H(\hat{\mathcal{G}}) = L(\mathcal{G}) + B$, where $B = \text{diag}(b_1 \ldots b_N) \in \mathbb{R}^{N\times N}$ with $b_i > 0$ when there is an edge from the leader to the i-agent and $L(\mathcal{G})$ is the Laplacian matrix associated to the communication topology of the followers.

Lemma 1. *[23] Let $\hat{\mathcal{G}}$ be the communication graph among all the agents with the leader as the root. Then, matrix $H(\hat{\mathcal{G}})$ (or simple H when the graph is clear from the context) is symmetric positive definite.*

B. On finite-time and fixed-time stability

Consider the system

$$\dot{x} = f(x; \vartheta), \qquad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, the vector $\vartheta \in \mathbb{R}^b$ stands for parameters of system (1) which are assumed to be constant, i.e., $\dot{\vartheta} = 0$. Furthermore, there is no limit for the number of parameters, so b can take any value in the natural number set $\mathbb{N}$. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear and the origin is assumed to be an equilibrium point of system (1), so that $f(0; \vartheta) = 0$.

Definition 1. *[24] The origin of (1) is globally finite-time stable if it is globally asymptotically stable and any solution $x(t; x_0)$ of (1) reaches the equilibrium point at some finite time moment i.e. $x(t; x_0) = 0, \forall t \geq T(x_0)$ where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is called the settling-time function.*

Definition 2. *[25] The origin of (1) is fixed-time stable if it is globally finite-time stable and the settling function is bounded, i.e., $\exists T_{\max} > 0 : T(x_0) \leq T_{\max}, \forall x_0 \in \mathbb{R}^n$.*

Theorem 1. *[26] Consider the system*

$$\dot{x} = -(\alpha\,|x|^p + \beta\,|x|^q)^k \text{sign}\,(x), \quad x(0) = x_0 \tag{2}$$

with $x \in \mathbb{R}$. The parameters of the system are the real numbers $\alpha, \beta, p, q, k > 0$ which satisfy the constraints $kp < 1$ and $kq > 1$. Let $\vartheta = [\alpha, \beta, p, q, k]^T \in \mathbb{R}^5$. Then, the origin $x = 0$ of system (2) is fixed-time stable and the settling time function satisfies $T(x_0) \leq T_f = \gamma(\vartheta)$, where

$$\gamma(\vartheta) = \frac{\Gamma\left(\frac{1-kp}{q-p}\right)\Gamma\left(\frac{qk-1}{q-p}\right)}{\alpha^k \Gamma(k)(q-p)} \left(\frac{\alpha}{\beta}\right)^{\frac{1-kp}{q-p}}, \tag{3}$$

and $\Gamma(\cdot)$ is the Gamma function defined as $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$.

Theorem 2. *[26] Consider the system*

$$\dot{x} = f(x; \rho), \quad x(0) = x_0 \tag{4}$$

where $x \in \mathbb{R}^n$ is the system state, the vector $\rho \in \mathbb{R}^b$ stands for the system parameters which are assumed to be constant. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that $f(0; \vartheta) = 0$. Assume *that there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$ and $V(x) > 0, \forall x \in \mathbb{R}^n \backslash \{0\}$ and the derivative of V along the trajectories of* (4) *satisfies*

$$\dot{V}(x) \leq -\frac{\gamma(\vartheta)}{T_c}(\alpha V(x)^p + \beta V(x)^q)^k, \forall x \in \mathbb{R}^n \backslash \{0\}$$

where $\alpha, \beta, p, q, k > 0$, $kp < 1$, $kq > 1$, γ is given in (3) *and $\dot{V}$ is the upper right-hand time-derivative of V. Then, the origin of* (4) *is predefined-time stable with predefined-time T_c.*

Definition 3. *For any real number r, the function $x \mapsto \lfloor x \rceil^r$ is defined as $\lfloor x \rceil^r = |x|^r \text{sign}(x)$ for any $x \in \mathbb{R}$ if $r > 0$, and for any $x \in \mathbb{R} \setminus 0$ if $r \leq 0$.*

III. Problem statement

Let us consider a group of $N + 1$ agents with one leader and N followers labeled 0 and $i \in \{1, \ldots, N\}$, respectively. The dynamics of the leader is described by

$$\begin{aligned} \dot{x}_0(t) &= v_0(t) \\ \dot{v}_0(t) &= u_0(t) \end{aligned}$$

where $x_0 \in \mathbb{R}$ is the position of the leader, $v_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}$ are respectively the velocity and acceleration of the leader, which is assumed to satisfy $|u_0(t)| \leq u_0^{max}, \forall t \geq 0$ with u_0^{max} a known constant. The dynamics of the $i-$th follower agent is given by:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= u_i(t) + \Delta_i(t) \end{aligned} \tag{5}$$

where $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the position, velocity and control of the $i-$th follower agent, and Δ_i is an unknown external disturbance which is assumed to satisfy $|\Delta_i(t)| \leq \delta, \forall t \geq 0$ with δ a known constant. Besides, each agent estimates the leader states, represented by $\hat{x}_i$ (position) and $\hat{v}_i$ (velocity). The communication topology is represented by an undirected graph, which is assumed to contain a spanning tree with the leader agent as the root. The $i-$th agent shares its estimated states of the leader with its neighbors, defined by the neighbor set $\mathcal{N}_i$.

The control objective is to design a distributed control u_i such that the consensus is achieved in a fixed-time T_c, i.e.,

$$\begin{cases} \lim_{t \to T_c} \|x_i(t) - x_0(t)\| = 0 \\ x_i(t) = x_0(t), \qquad \forall t > T_c. \end{cases}$$

To achieve this goal, the control scheme is the following. Due that the leader velocity is not available, and only a subset of agents has access to the leader position, a differentiator is implemented in the subset of agents with access to the leader position to estimate the velocity. Then, a distributed observer is implemented in each agent to estimate the leader states. Finally, after the observer converges, a controller drives the agent's states towards the leader's trajectory.

IV. Proposed solution

To estimate the leader's states in the i-th agent and when the velocity is not available, we use the following predefined-time exact differentiator.

Proposition 1. *(predefined-time differentiator) [27] Let u_0 be the control input of the leader, which is assumed to satisfy $|u_0| \leq u_0^{max}$ and let*

$$\begin{aligned} g_1(w) &= 4\sqrt{L}(\lfloor w \rceil^{\frac{1}{2}} + k\lfloor w \rceil^{\frac{3}{2}}) \\ g_2(w) &= 2L(\text{sign}(w) + 4k^2 w + 3k^4 \lfloor w \rceil^2) \end{aligned}$$

where $L = u_0^{max}$ and $k = \frac{9.8}{\sqrt{L}}$. Then, choosing $T_{c_1} > 0$ and $\hat{\alpha} > 0$, and using the algorithm

$$\begin{aligned} \dot{z}_1 &= -\varrho(t)(-\hat{\alpha}e_1 + \hat{\beta}g_1(\hat{\beta}^{-1}e_1)) + z_2 \\ \dot{z}_2 &= -\varrho(t)^2(\hat{\alpha}^2 e_1 + \hat{\beta}g_2(\hat{\beta}^{-1}e_1) - \hat{\alpha}\hat{\beta}g_1(\hat{\beta}^{-1}e_1)) \end{aligned} \tag{6}$$

for $t \in [0, \big(1 - \exp(-\hat{\alpha})\big)T_{c_1})$ and

$$\begin{aligned} \dot{z}_1 &= -g_1(e_1) + z_2 \\ \dot{z}_2 &= -g_2(e_1) \end{aligned}$$

for $t \geq \big(1 - \exp(-\hat{\alpha})\big)T_{c_1}$; where $e_1 = z_1 - x_0$, $\hat{\beta} = (\hat{\alpha}T_{c_1})^2$ and

$$\varrho(t) = \frac{1}{\hat{\alpha}(T_c - t)},$$

Then, for all $t \geq T_{c_1}$ it holds that $z_1(t) = x_0$ and $z_2(t) = v_0$.

Proof. This predefined-time differentiator can be obtained with the redesign methodology proposed in [27] using [28] as a base algorithm. □

Considering that the leader position is not available for all followers, an observer is designed for each agent to estimate the state of the leader in a fixed-time. The observer has the following structure:

$$\begin{aligned} \dot{\hat{x}}_i =& \hat{v}_i - \kappa_{i,x} \left[(\alpha|e_{1,i}|^p + \beta|e_{1,i}|^q)^k + \zeta_x\right] \text{sign}(e_{1,i}) \\ \dot{\hat{v}}_i =& -\kappa_{i,v} \left[(\alpha|e_{2,i}|^p + \beta|e_{2,i}|^q)^k + \zeta_v\right] \text{sign}(e_{2,i}) \end{aligned} \tag{7}$$

with $e_{1,i} = \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{x}_j - \hat{x}_i) + b_i(x_0 - \hat{x}_i)$ and $e_{2,i} = \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{v}_j - \hat{v}_i) + b_i(z_2 - \hat{x}_i)$, where $\hat{x}_i$ (resp. $\hat{v}_i$) is the estimate of the leader position (resp. velocity) for the i-th follower, and z_2 is the estimated velocity of the leader from (6). For each agent, let us denote the observer errors as

$$\begin{aligned} \tilde{x}_i &= \hat{x}_i - x_0 \\ \tilde{v}_i &= \hat{v}_i - v_0, \end{aligned}$$

where v_0 is estimated by (6) and $\dot{v}_0$ is unknown, which is assumed to satisfy $|\dot{v}_0| \leq u_0^{max}$, $\forall t \geq 0$ with u_0^{max} a known constant and $|\frac{dz_2}{dt}| \leq u_0^{max}$. Thus, the observation error dynamics can be expressed as:

$$\begin{aligned} \dot{\tilde{x}}_i &= \tilde{v}_i - \kappa_{i,x} \left[(\alpha|e_{1,i}|^p + \beta|e_{1,i}|^q)^k + \zeta_x\right] \text{sign}(e_{1,i}) \\ \dot{\tilde{v}}_i &= -\kappa_{i,v} \left[(\alpha|e_{2,i}|^p + \beta|e_{2,i}|^q)^k + \zeta_v\right] \text{sign}(e_{2,i}) - \dot{v}_0 \end{aligned} \tag{8}$$

with $e_{1,i} = \sum_{j \in \mathcal{N}_i} a_{ij}(\tilde{x}_j(t) - \tilde{x}_i(t)) - b_i\tilde{x}_i(t))$ and $e_{2,i} = \sum_{j \in \mathcal{N}_i} a_{ij}(\tilde{v}_j(t) - \tilde{v}_i(t)) - b_i\tilde{v}_i(t))$. In a compact form, with

$\tilde{x} = [\tilde{x}_1 \ \cdots \ \tilde{x}_N]^T \in \mathbb{R}^N$ and $\tilde{v} = [\tilde{v}_1 \ \cdots \ \tilde{v}_N]^T \in \mathbb{R}^N$, the system (8) can be written as:

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{v} - \Phi_x (H\tilde{x}) \\ \dot{\tilde{v}} &= -\Phi_v (H\tilde{v}) - \mathbf{1}\dot{v}_0 \end{aligned} \tag{9}$$

where $\mathbf{1} = [1, \cdots, 1]^T \in \mathbb{R}^N$, H is defined as in Lemma 1, and the function $\Phi_x : \mathbb{R}^N \to \mathbb{R}^N$ is defined as

$$\Phi_x(y) = \begin{bmatrix} \kappa_{1,x} \left[(\alpha|y_1|^p + \beta|y_1|^q)^k + \zeta_x\right] \mathrm{sign}\,(y_1) \\ \vdots \\ \kappa_{N,x} \left[(\alpha|y_N|^p + \beta|y_N|^q)^k + \zeta_x\right] \mathrm{sign}\,(y_N) \end{bmatrix},$$

and respectively for $\Phi_v : \mathbb{R}^N \to \mathbb{R}^N$.

Theorem 3. *If the observer parameters are selected as* $\alpha, \beta, p, q, k > 0$, $kp < 1$, $kq > 1$, $\zeta_x \geq 0$,

$$\kappa_x \geq \frac{N\gamma(\vartheta)}{\lambda_{\min}(H)T_{o_2}},\ \kappa_v \geq \frac{N\gamma(\vartheta)}{\lambda_{\min}(H)T_{o_1}} \text{ and } \kappa_v\zeta_v \geq u_0^{max}$$

where $\kappa_x = \min_{i\in\{1\ldots N\}} \kappa_{i,x}$ *and* $\kappa_v = \min_{i\in\{1\ldots N\}} \kappa_{i,v}$, *and* $\gamma(\vartheta)$ *is defined in Equation* (3)*, then under the distributed observer* (7)*, the observer error dynamics* (8) *is fixed-time stable with a predefined-time* $T_o = T_{o_1} + T_{o_2}$.

Proof. Consider the radially unbounded Lyapunov function candidate

$$V_1(\tilde{v}) = \frac{1}{N}\sqrt{\lambda_{\min}(H)\,\tilde{v}^T H\tilde{v}}.$$

Its time-derivative along the trajectories of system (9), with $e_2 = H\tilde{v} = [e_{2,1} \ \cdots \ e_{2,N}]^T$, can be written as follows

$$\begin{aligned} \dot{V}_1 &= \frac{\sqrt{\lambda_{\min}(H)}}{N}\frac{e_2^T}{\sqrt{\tilde{v}^T H\tilde{v}}}\left(-\Phi_v(H\tilde{v}) - \mathbf{1}\dot{v}_0\right) \\ &= \frac{\sqrt{\lambda_{\min}(H)}}{N\sqrt{\tilde{v}^T H\tilde{v}}}\left(-\sum_{i=1}^{N}\kappa_{i,v}\,|e_{2,i}|\left[(\alpha|e_{2,i}|^p + \beta|e_{2,i}|^q)^k\right]\right. \\ &\qquad \left. - \sum_{i=1}^{N}\kappa_{i,v}\,|e_{2,i}|\,\zeta_v - e_2^T(\mathbf{1}\dot{v}_0)\right) \end{aligned} \tag{10}$$

Now, for the last two terms of Equation (10), one can obtain

$$-\zeta_v\sum_{i=1}^{N}\kappa_{i,v}\,|e_{2,i}| - e_2^T(\mathbf{1}\dot{v}_0) \leq -\|e_2\|_1\left(\kappa_v\zeta_v - u_0^{max}\right) \leq 0.$$

From [15], the first terms of (10) can be written as

$$\begin{aligned} &\frac{1}{\sqrt{\tilde{v}^T H\tilde{v}}}\sum_{i=1}^{N}\kappa_{i,v}\,|e_{2,i}|\,[\alpha|e_{2,i}|^p + \beta|e_{2,i}|^q]^k \\ &\qquad \geq \kappa_v\sqrt{\lambda_{\min}(H)}\left(\alpha V_1^p + \beta V_1^q\right)^k. \end{aligned}$$

Therefore, the following inequality can be obtained

$$\dot{V}_1 \leq -\frac{\kappa_v\lambda_{\min}(H)}{N}\left(\alpha V_1^p + \beta V_1^q\right)^k.$$

From Theorem 2, the observation error in velocity $\tilde{v}$ converges to the origin in a fixed-time before the predefined-time T_{o_1} where $\gamma(\vartheta)$ is given by Equation (3).

Once the observation error in velocity $\tilde{v}$ converges to zero (i.e. after time T_{o_1}), the observation error dynamics in position can be reduced to

$$\dot{\tilde{x}}_i = -\kappa_{i,x}\left[(\alpha|e_{1,i}|^p + \beta|e_{1,i}|^q)^k + \zeta_x\right]\mathrm{sign}\,(e_{1,i}).$$

Similarly to the previous analysis, one can easily show that

$$V_2(\tilde{x}) = \frac{1}{N}\sqrt{\lambda_{\min}(H)\,\tilde{x}^T H\tilde{x}}$$

satisfies

$$\dot{V}_2(\tilde{x}) \leq -\frac{\kappa_x\lambda_{\min}(H)}{N}\left(\alpha V_2^p + \beta V_2^q\right)^k, \forall t \geq T_{o_2}.$$

From Theorem 2, the observation error in position $\tilde{x}$ converges to the origin in a fixed-time before the predefined-time T_{o_2}.

Therefore, the proposed distributed observer guarantees the estimation of the leader position (x_0) and the estimated leader velocity (z_2) in a fixed-time before the predefined-time $T_o = T_{o_1} + T_{o_2}$. Moreover, when $\omega_1 = 0$, $\forall t > T_{c_1}$ and $z_2 = \frac{dx_0}{dt}$ $\forall t > T_{c_1}$ with $T_{c_1} < T_o$, the distributed observer with the differentiator guarantee the estimation of the leader states. □

In a distributed way, each agent has an accurate estimation of the leader state. Then, for each agent, the tracking errors are defined as $e_{x,i} = x_i - x_0$ and $e_{v,i} = v_i - \hat{v}_0$ after the convergence of the observation error. Then, its dynamics can be expressed as:

$$\begin{aligned} \dot{e}_{x,i} &= e_{v,i} \\ \dot{e}_{v,i} &= u_i + \Delta_i. \end{aligned}$$

Here, the objective is to design the control input u_i such that the origin $(e_{x,i}, e_{v,i}) = (0, 0)$ is fixed-time stable where the *UBST* is set a priory by the user, in spite of the unknown but bounded perturbation term Δ_i. Then, the following controller from [26] is implemented in each agent to achieve the leader follower consensus.

Theorem 4. *[26] If for each agent, the controller is selected as*

$$\begin{aligned} u_i = -\Bigg[&\frac{\gamma_2}{\hat{T}_{c_2}}\left(\alpha_2\,|\sigma_i|^{p'} + \beta_2\,|\sigma_i|^{q'}\right)^{k'} + \\ &\frac{\gamma_1^2}{2\hat{T}_{c_1}^2}\left(\alpha_1 + 3\beta_1 e_{x,i}^2\right) + \zeta_i(t)\Bigg]\mathrm{sign}\,(\sigma_i) \end{aligned} \tag{11}$$

with the following sliding variable

$$\sigma_i = e_{v,i} + \left\lfloor \lfloor e_{v,i}\rceil^2 + \frac{\gamma_1^2}{\hat{T}_{c_1}^2}\left(\alpha_1\,\lfloor e_{x,i}\rceil^1 + \beta_1\,\lfloor e_{x,i}\rceil^3\right)\right\rceil^{1/2}, \tag{12}$$

where parameters are selected as $\alpha_1, \alpha_2, \beta_1, \beta_2, \hat{T}_{c_1}, \hat{T}_{c_2} > 0$, $p', q', k' > 0$, $k'p' < 1$, $k'q' > 1$, $\zeta_i \geq u_0^{max} + \delta_i$*, and the functions* γ_1, γ_2 *are taken from [26]. Then, the leader-follower consensus is achieved in a predefined-time* $\hat{T} = t_0 + \hat{T}_{c_1} + \hat{T}_{c_2}$.

Proof. Let us consider the candidate Lyapunov function $V_1(\sigma_i) = |\sigma_i|$ with its time derivative as $\dot{V}_1 = \text{sign}(\sigma_1)\,\dot{\sigma}_i$. Then from (12), the Lyapunov function derivative is rewritten, using (11), as follows

$$\dot{V}_1(\sigma) \leq -\frac{\gamma_2}{\hat{T}_{c_2}}\left(\alpha_2 V_1(\sigma)^{p'} + \beta_2 V_1(\sigma)^{q'}\right)^{k'}.$$

From Theorem 2, one can deduce that σ_i converges to zero in a fixed-time $\hat{T}_{c_2}$. After time $\hat{T}_{c2}$, one obtains

$$\dot{e}_{x,i} = e_{v,i} = -\frac{\gamma_1}{\hat{T}_{c_1}}\left(\alpha_1 |e_{x,i}| + \beta_1 |e_{x,i}|^3\right)^{1/2} \text{sign}(e_{x,i}).$$

From Theorem 1, it is clear that $e_{x,i}$ converges to zero in a fixed-time before the settling time $\hat{T}_{c_1}$. Moreover, from (12), since $\sigma_i = 0$ and $e_{x,i} = 0$, then $e_{v,i} = 0$. Hence, we can conclude that system (5) with (11) as the control input, is fixed-time stable with predefined-time $\hat{T}_{c_1} + \hat{T}_{c_2}$. □

V. Simulation results

In this section, we illustrate the application of the main results of this paper, i.e., the differentiator and the proposed observer with a control which provides fixed-time convergence.

For the example, consider a multi-agent system composed of $N = 5$ agents where the dynamics of each agent is given by Equation (5). The communication topology is shown in Fig. 1, which is an undirected graph and contains a spanning tree with the leader as the root. For the leader, its velocity (v_o) is given by $v_0 = -0.4\cos(t) - 0.64\sin(0.8t)$ with the initial conditions $[x_0(0), v_0(0)] = [0.8, -0.4]$ and the perturbation for each follower is given by $\Delta_i(t) = \sin(40t + 0.5i)$. From Fig. 1, one gets $\lambda_{\min}(H) = 0.1392$. The initial conditions of the follower agents are as follows $x(0) = [13.6, 18.8, 18.2, -0.50, 12.0]$, $v(0) = [-14.3, -3.1, 16.6, 11.6, 18.3]$, and the initial conditions for the observer are set arbitrarily as $\hat{x}(0) = [27.2, 37.6, 36.4, -1, 24]$, $\hat{v}(0) = [-28.6, -6.2, 33.2, 23.2, 36.6]$. In order to estimate the leader velocity, the auxiliary system (6) from Proposition 1 is implemented in the set of agents with access to the leader position, where $\hat{\alpha} = 2.5$ and $T_{c_1} = 0.5$. Fig. 2 shows the states of the auxiliary system (6), where it is possible to see that the auxiliary system (red line) reaches the leader states (doted line) before the predefined-time T_{c_1}. Now, according to Theorem 3, the distributed fixed-time observer (7) with $p = 1.5$, $q = 3.0$, $k = 0.5$, $\alpha = 1$, $\beta = 2$, $T_{o_1} = 0.9s$, $T_{o_2} = 0.1s$, $\kappa_x = 179.496$, $\kappa_v = 1794.9$, $\zeta_x = 0.1046$ and $\zeta_v = 0.0161$, guarantees that the observer error converges to zero before the predefined-time $T_{o_1} + T_{o_2} = 1s$. In order to follow the trajectory of the leader, the parameters of the Theorem 4 are $p' = 1.5$, $q' = 3.0$, $k' = 0.5$, $\alpha_1 = \alpha_2 = 1/\beta_1 = 1/\beta_2 = 1/4$, $\hat{T}_{c_1} = 1s$ and $\hat{T}_{c_2} = 1s$. Fig. 3 shows the leader states estimation of each agent (continuous lines) and the leader states (doted line), where it can be seen that the observer converges to the leader states before the predefined-time T_o. Finally, Fig. 4 shows the trajectories of each agent, where each trajectory converges to the leader trajectory in a fixed-time before the predefined time.

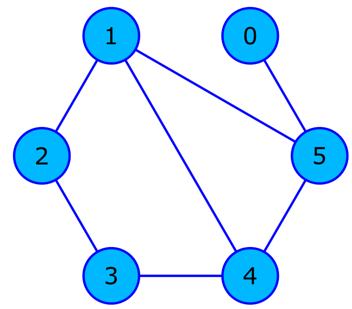

Fig. 1. Communication topology with 5 followers.

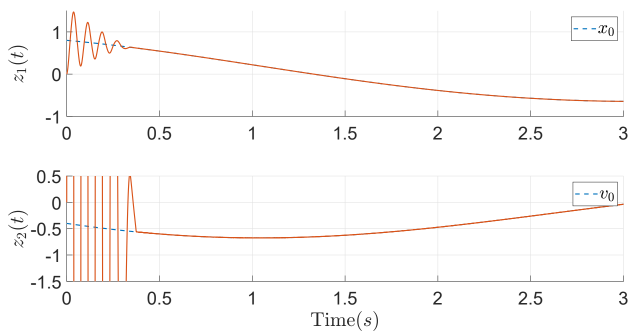

Fig. 2. Differentiation of the leader state

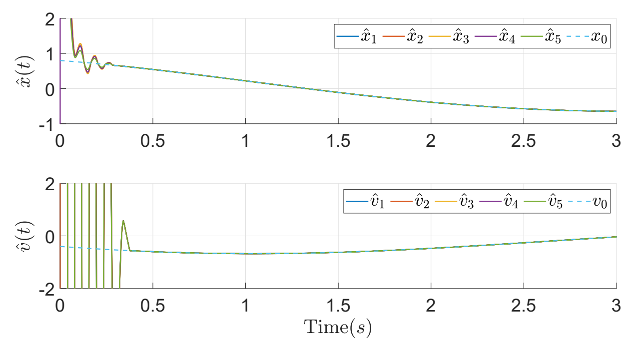

Fig. 3. Leader states estimation for each agent

VI. Conclusions

In this paper, we present a novel protocol to solve the problem of the leader-follower consensus, where only partial information of the leader states is available. Then, from available information, the rest of the states is estimated in a predefined-time which is established a priory. Once the leader state is available for a subset of followers, a distributed observer estimates in a fixed-time the leader states. Finally, a controller is proposed to drive the agents states to the estimated states. The proposed protocol allows to establish a priory the convergence time, which can be adjusted according to the

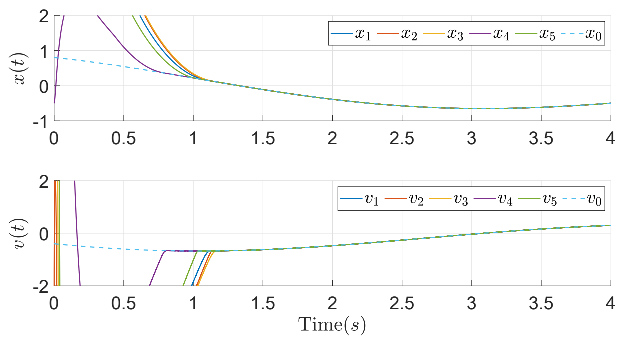

Fig. 4. Trajectories for each agent

constraints of the system. Future work includes an observer using only partial information of the neighbors when the communication is limited.

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